Ramsey Theory

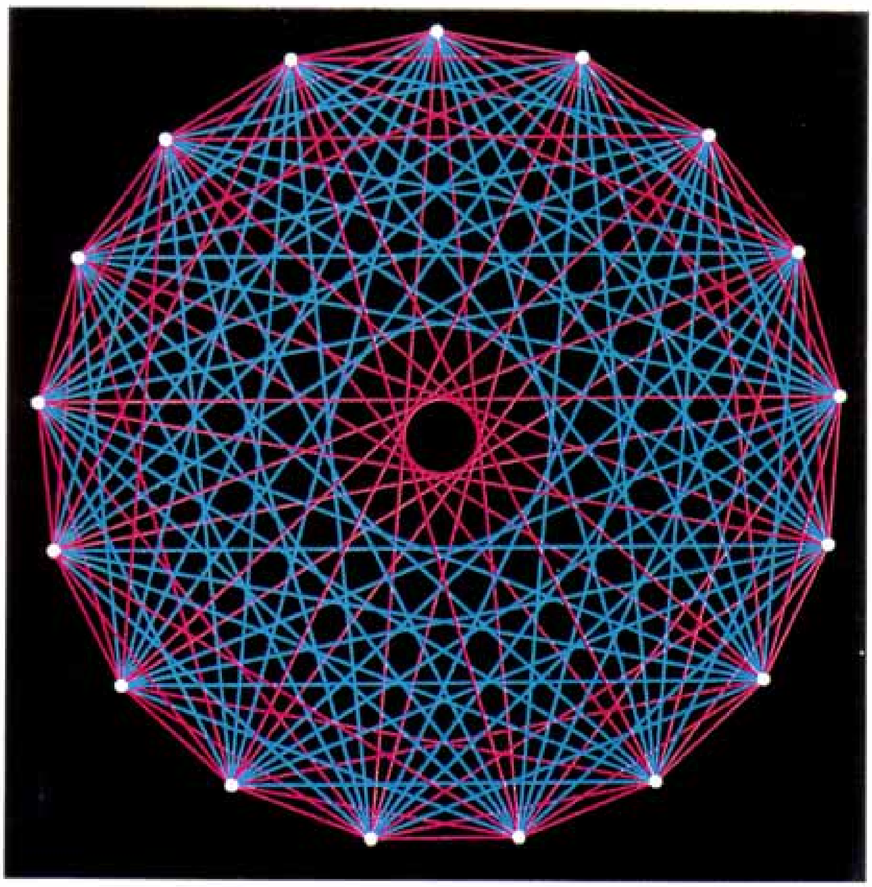
The brilliant mathematician Frank Plumpton Ramsey proved that complete disorder is an impossibility. Every large set of numbers, points or objects necessarily contains a highly regular pattern

by Ronald L. Graham and Joel H. Spencer

According to a 3,500-year-old cuneiform text, an ancient Sumerian scholar once looked to the stars in the heavens and saw a lion, a bull and a scorpion. A modern astronomer would be more likely to describe a constellation as a temporary collection of stars, which we earthlings observe from the edge of an ordinary galaxy. Yet most stargazers would agree that the night sky appears to be filled with constellations in the shape of straight lines, rectangles and pentagons. Could it be that such geometric patterns arise from unknown forces in the cosmos?

Mathematics provides a much more plausible explanation. In 1928 Frank Plumpton Ramsey, an English mathematician, philosopher and economist, proved that such patterns are actually implicit in any large structure, whether it is a group of stars, an array of pebbles or a series of numbers generated by throws of a die. Given enough stars, for instance, one can always find a group that very nearly forms a particular pattern: a straight line, a rectangle or, for that matter, a big dipper. In fact, Ramsey theory states that any structure will necessarily contain an orderly substructure. As the late American mathematician Theodore S. Motzkin first proclaimed some 25 years ago, Ramsey theory implies that complete disorder is an impossibility.

Ramsey theorists struggle to figure out just how many stars, numbers or figures are required to guarantee a certain desired substructure. Such problems often take decades to solve and yield to only the most ingenious and delicate reasoning. As Ramsey theorists search for solutions, they assist engineers attempting to build better communications networks as well as information transmission and retrieval systems. Ramsey theorists have also discovered some of the mathematical tools that will guide scientists in the next century. Perhaps most important, Ramsey theorists are probing the ultimate structure of mathematics, a structure that transcends the universe.



PARTY PUZZLE typifies the problems that Ramsey theory addresses. How many people does it take to form a group that always contains either four mutual acquaintances or four mutual strangers? In the diagram, points represent people. A red edge connects people who are mutual acquaintances, and a blue edge joins people who are mutual strangers. In the group of 17 points above, there are no four points whose network of edges are either completely red or completely blue. Therefore, it takes more than 17 people to guarantee that there will always be four people who are either acquaintances or strangers. In fact, in any group of 18 people, there are always either four mutual acquaintances or four mutual strangers.

Unlike many branches of mathematics that interest professionals today, Ramsey theory can be presented intuitively. Indeed, the charm of Ramsey theory is derived in part from the simplicity with which the problems can be stated. For example, if six people are chosen at random (say, Alfred, Betty, Calvin, Deborah, Edward and Frances), is it true that either three of them mutually know one another or three of them mutually do not know one another?

We can solve the "party puzzle" in many ways. We could list all conceiv-

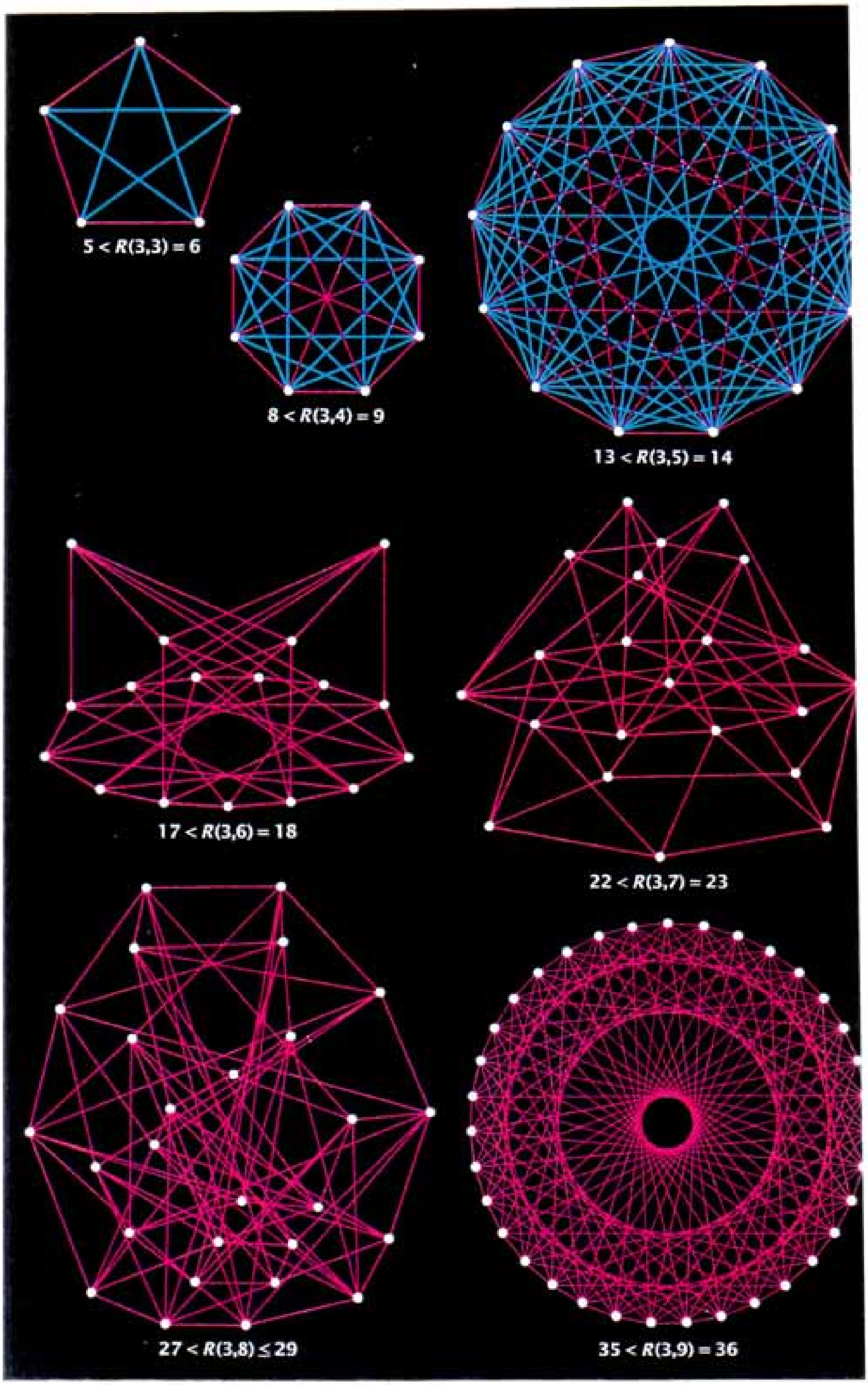
RONALD L. GRAHAM and JOEL H. SPENCER have written the definitive text on Ramsey theory with Bruce L. Rothschild of the University of California, Los Angeles. Graham is adjunct director for research of the information sciences division at AT&T Bell Laboratories and professor of mathematics at Rutgers University. In 1962 he received his Ph.D. in mathematics from the University of California, Berkeley. This is Graham's third article for *Scientific American*. Spencer is professor of mathematics and computer science at New York University's Courant Institute of Mathematical Sciences. In 1970 he earned his Ph.D. in mathematics from Harvard University. Spencer has a particular interest in the history of Hungarian mathematics, such as the work of Paul Erdős, who pioneered much of Ramsey theory.

able combinations and check each one for an acquainted or unacquainted group of three. But because we would have to check 32,768 (or 2^{15}) combinations, this brute-force method is neither practical nor insightful.

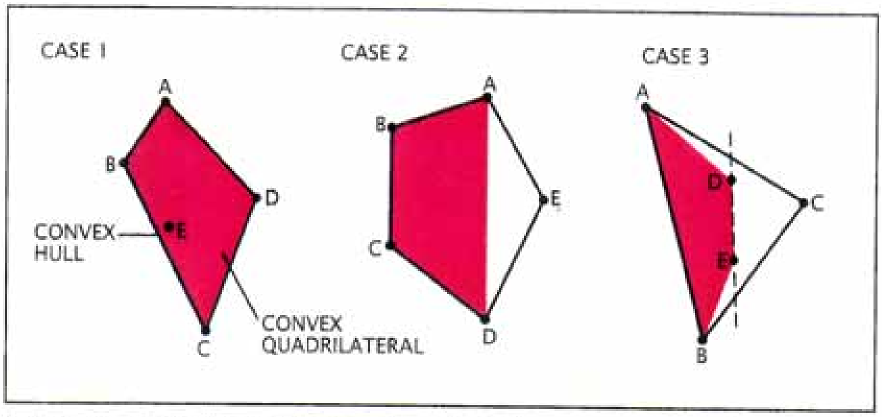
Fortunately, we can find the answer by considering two simple cases. In the first case, suppose Alfred knows three (or more) of the others, say, Betty, Calvin and Deborah. If either Betty and Calvin or Betty and Deborah or Calvin and Deborah are mutual acquaintances, then Alfred and the acquainted pair make three people who know one another. Otherwise Betty, Calvin and Deborah are mutual strangers. In the second case, suppose Alfred knows only two (or fewer) of the others, say, Betty and Calvin. If either Deborah and Edward or Deborah and Frances or Edward and Frances are strangers, then Alfred and the unacquainted pair make three people who do not know one another. Otherwise Deborah, Edward and Frances are mutual acquaintances. In just six sentences, we have shown why any party of six people must include at least three mutual acquaintances or three mutual strangers. More to the point, the solution to the party puzzle is a special case of Ramsey theory.

By generalizing this special case, we can give the full theorem. Instead of considering six people in the problem, we can have any number of people or, for that matter, any number of objects. We need not restrict ourselves to two relationships, acquaintances and strangers. We can have any number of mutually exclusive relationships—for instance, friends, foes and neutral parties.

We can then describe the full Ramsey theorem. If the number of objects in a set is sufficiently large and each pair of



RAMSEY NUMBERS are defined as the smallest value of n such that in a group of n points either a group of j points forms a complete network of red edges or a group of k points forms a complete network of blue edges. The diagrams above indicate how large a particular Ramsey number should be. The first diagram shows five points connected by red and blue edges in such a way that no three points form either a red or a blue complete network. Hence, the first diagram implies that the Ramsey number for red three and blue three must be greater than five. In a similar manner, one can argue that the second diagram suggests that the Ramsey number for red three and blue four is greater than eight. By other more complicated techniques, it can be demonstrated that the Ramsey number for red three and blue three is six and that the number for red three and blue four is nine. All of the exact Ramsey numbers that are known are given above, except the Ramsey number for red four and blue four, whose diagram is shown on the opposite page. (In some of the diagrams the blue edges are omitted for simplicity.) The Ramsey number for red three and blue eight has been proved to be greater than 27 and less than or equal to 29. Recently it was shown (but not yet verified) to be 28.



RAMSEY THEORY was rediscovered in 1933 when a young student, Esther Klein, introduced a geometric problem: if five points lie in a plane so that no three points form a straight line, prove that four of the points will always form a convex quadrilateral. All cases of the problem are variations of the three above. The simplest case occurs when the convex hull—the convex polygon enclosing all the points—is a quadrilateral. If the convex hull is a pentagon, then any four points can be connected to form a quadrilateral. A triangular convex hull will always contain two points, here *D* and *E*. The line *DE* splits the triangle so that two points, *A* and *B*, are on one side. The four points *ABDE* must form a convex quadrilateral.

objects has one of a number of relations, then there is always a subset containing a certain number of objects where each pair has the same relation.

Frank Ramsey, who first proved this statement in 1928, grew up in Cambridge, England. His father, Arthur S. Ramsey, was professor of mathematics and president of Magdalene College at Cambridge. In 1925 young Ramsey graduated as the university's top mathematics student. Although philosophy and mathematical logic chiefly engaged him, he also contributed to economics, probability, decision theory, cognitive psychology and semantics.

Shortly after graduation, he joined a group of economists headed by John Maynard Keynes. Ramsey wrote only two papers on mathematical economics, but both are still widely cited. In philosophy his inspiration came from George E. Moore, Ludwig Wittgenstein and Bertrand Russell. Moore wrote, "He was an extraordinarily clear thinker: no one could avoid more easily than he the sort of confusions of thought to which even the best philosophers are liable." Then, tragically, in 1930, at the age of 26, Ramsey took ill and died of complications from abdominal surgery.

An irony is attached to the story of how, two years before his death, Ramsey derived his eponymous theory. He came on the central idea while attempting to prove a premise put forward by Russell and Alfred North Whitehead in their masterwork, *Principia Mathematica*. They proposed that all mathematical truths can be deduced from a concise set of axioms. Expanding on their idea, the German mathematician David Hilbert had suggested that there must be a procedure to decide whether or not a given proposition follows from a particular set of axioms. Ramsey showed that there was such a decision procedure for a special case. (A few years later Kurt Gödel, followed by the English mathematician Alan M. Turing and others, showed conclusively that for the general case, there was no such decision procedure.)

Ramsey proved his theorem as a first step in his attempt to demonstrate the special case. As it turned out, he could have accomplished the same task by other means. Ramsey had proved a theorem that was superfluous to an argument, which he could never have proved in the general case.

There matters lay until 1933, when two Hungarian mathematicians, Paul Erdős and George Szekeres, rediscovered Ramsey theory. They are largely responsible for its popularization in the mathematics community. At the time, Erdős was a 19-year-old student at the University of Budapest, and Szekeres had recently earned a degree in chemical engineering from the Technical University of Budapest. They and a group of fellow students would meet almost every Sunday in a park or at school, mainly to discuss mathematics.

At a meeting during the winter of 1933, one of the students, Esther Klein, challenged the group to solve a curious problem: if five points lie in a plane so that no three points form a straight line, prove that four of the points will always form a convex quadrilateral. (The term convex suggests a bulging geometric figure such as a hexagon but not a five-pointed star. More specifically, a polygon is convex if every line segment drawn between its vertices lies inside the polygon.)

After allowing her friends to contemplate the problem, Klein presented a proof [*see illustration at left*]. Erdős and Klein quickly came up with a generalization of the problem. They realized that five of nine points in a plane will always form a convex pentagon. They then offered a new problem: if the number of points that lie in a plane is equal to $1+2^{k-2}$ where k is 3 or 4 or 5 and so on, can one always select k points so that they form a convex k-sided polygon?

In a memoir Szekeres recalled the scene: "We soon realized that a simple-minded argument would not do and there was a feeling of excitement that a new type of geometrical problem emerged from our circle." Szekeres eagerly demonstrated that there always exists a number n such that if n points lie in the plane so that no three form a straight line, it is possible to select k points that form a convex k-sided polygon. In other words, given enough points, one can always find a set that will form a particular polygon. In proving this, Szekeres had rediscovered Ramsey's theorem, although no one in the group knew it at the time.

In 1934 Erdős and Szekeres reported their results, but neither they nor anyone else to this day has been able to prove Erdős's conjecture that precisely $n=1+2^{k-2}$ points suffice. Erdős often refers to their joint publication as the "Happy End Paper," because soon after publication Szekeres and Klein married. Erdős became the most prolific mathematician of this century.

Erdős was intrigued by Ramsey's idea that any sufficiently large structure must contain a regular substructure of a given size. But he wondered exactly how large the structure must be to guarantee a certain substructure. So Erdős began work on a version of the party puzzle.

In this version the six people are represented as six points. For convenience, the points are drawn on a plane so that no three are in a line. The points are connected by an edge, which is colored to represent the relationship of the corresponding two people. A red edge means the people mutually know one another, and a blue edge means they do not know one another.

Hence, if three people are mutual acquaintances, the edges between the points will form a red triangle, and if three people are mutual strangers, a blue triangle will be formed. The party puzzle can be rephrased as follows: Is it true that if each edge between six points is arbitrarily colored either red or blue, there will always be either a red or a blue triangle?

The problem that Erdős studied is a general version of this one. He defined a complete network as a number of points that are all connected by edges. He then asked what the smallest complete network is that when arbitrarily colored red and blue will guarantee either a red or a blue complete network of three points. The answer is a complete network of six points. The problem and solution can be more conveniently expressed as follows: the Ramsey number for red three and blue three is equal to six.

But what about the Ramsey number for red five and blue three? In other words, what is the smallest complete network that when arbitrarily colored red and blue will guarantee either a red network of five points or a blue network of three points? The Ramsey number for red five and blue three is 14, which was not proved until 1955 by Robert E. Greenwood of the University of Texas at Austin and Andrew M. Gleason of Harvard University.

Ramsey numbers are notoriously difficult to calculate. The efforts of generations of mathematicians and computers have succeeded in finding only seven Ramsey numbers, which are given in the illustration on page 113.

To express the difficulty of calculating Ramsey numbers, Erdős often tells the following anecdote. Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshall the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Erdős did find a way, however, to get some idea of how large a Ramsey number must be. What if he could find a red and blue coloring of a large complete network that did *not* form either a red or a blue network of three points? Such a coloring of the complete network of five points is shown on page 113. It follows that the Ramsey number for red three and blue three must be greater than five. Five is a lower bound for that Ramsey number.

In 1947 Erdős proposed an unusual method for finding the lower bound of any Ramsey number: flip a coin. He performed a thought experiment in which each edge of a complete network of, say, a million points was colored according to the flip of a fair coin. The edge would be red for "tails" or blue for "heads." He then tried to show that the Ramsey number for, say, red 34 and blue 34 is greater than a million. The experiment was a success if there was neither a red nor a blue network of 34 points.

How could he ensure success? Any 34 points must have 561 edges between them. If the first coin flip specifies blue for the first edge, then the next 560 flips must also specify blue in order to produce a blue network. The probability of this occurring is one half to the power of 561. A red network has an equal probability of occurring so the total probability is double, or approximately 2.6×10^{-169}.

Now the number of sets of 34 points in a million points is equal to $(1{,}000{,}000 \times 999{,}999 \times \ldots \times 999{,}967)$ divided by $(34 \times 33 \times \ldots \times 2 \times 1)$, which is about 3.4×10^{165}. Out of all possible complete networks of 34 points, therefore, one would expect $(3.4 \times 10^{165}) \times (2.6 \times 10^{-169})$, or roughly .001, to be monochromatic. Thus, 99.9 percent of the time the thought experiment is a success with no 34-point monochromatic sets created.

Erdős then employed a subtle reductio ad absurdum. He hypothesized that no coloring scheme was successful. Then the thought experiment would have zero probability of success, which he knew was not the case. The hypothesis must be incorrect; there *must* be a successful coloring (not just with 99.9 percent certainty but with absolute certainty). The existence of a coloring implies that one million is a lower bound for the Ramsey number for red 34 and blue 34.

This argument, which is known as the probabilistic method, has given the best available lower bounds for Ramsey numbers. The probabilistic method gives no clues, however, concerning the actual "construction" of the desired coloring. In an attempt to produce these colorings, workers have applied a range of techniques from number theory, set theory and other branches of mathematics. The results, although interesting, do not yet approach the bounds obtained by flipping a coin.

Although much of the early work in Ramsey theory focused on sets of points and lines, many of the first problems involved sets of numbers. In fact, the Dutch mathematician Bartel L. van der Waerden began solving such problems before Ramsey proved his theorem.

In 1926 van der Waerden learned

Ramsey Theory and Arithmetic Progressions

An arithmetic progression is a sequence of numbers in which the difference between successive terms remains constant. For instance, 7, 10, 13, 16 is an arithmetic progression in which the difference between successive terms is three. The following statement about arithmetic progressions follows from Ramsey theory: if each number 1 through 9 is colored either red or blue, either three red or three blue numbers will form an arithmetic progression.

To prove the conjecture, we might check every one of the 512 ways that nine numbers can be colored. Yet we can prove the conjecture by considering only two cases. We start with the case in which 4 and 6 are the same color, say, blue.

1 2 3 4 5 6 7 8 9

To avoid the red arithmetic progression 4, 5, 6, we color 5 red.

1 2 3 4 5 6 7 8 9

To avoid the blue arithmetic progressions 2, 4, 6 and 4, 6, 8, we color 2 and 8 red.

1 2 3 4 5 6 7 8 9

But that leaves the red arithmetic progression 2, 5, 8. Hence, if 4 and 6 are the same color, there is always either a red or a blue arithmetic progression. Next we consider the case in which 4 and 6 are different colors. We can color 5 either red or blue without forming an arithmetic progression, so we arbitrarily choose to color 5 red.

1 2 3 4 5 6 7 8 9

We continue to color the numbers as follows:

3 to avoid 3 4 5
7 to avoid 5 7 9
2 to avoid 2 5 8
9 to avoid 3 6 9
8 to avoid 6 7 8
1 to avoid 1 2 3

This coloring yields the sequence

1 2 3 4 5 6 7 8 9

Yet we are still left with the red arithmetic progression 1, 5, 9. Therefore, regardless of whether 4 and 6 are the same color or different, there is always either a red or a blue arithmetic progression.

about a curious problem involving arithmetic progressions. As the phrase implies, an arithmetic progression is a sequence of numbers in which the difference between successive terms remains constant. For example, the sequence 3, 5, 7 is a three-term arithmetic progression in which the difference between successive terms is two. A special case of the problem that caught van der Waerden's interest was the following: If each integer from 1 through 9 is printed on a page in one of two colors, either red or blue, is it always true that either red three or blue three numbers will form an arithmetic progression? The answer is given in the box on the preceding page.

Van der Waerden challenged himself with the following generalization: if *n* is a sufficiently large integer and if each integer from 1 through *n* is printed arbitrarily in one of two colors, then there is always a monochromatic arithmetic progression with a certain number of terms. One can think of this statement as Ramsey's theorem for arithmetic progressions, although it is generally known as van der Waerden's theorem.

Van der Waerden enlisted the help of his colleagues Emil Artin and Otto Schreier. He later wrote: "We went into Artin's office in the Mathematics Department of the University of Hamburg, and tried to find a proof. We drew some diagrams on the blackboard. We had what the Germans call 'Einfälle': sudden ideas that flash into one's mind. Several times such new ideas gave the discussion a new turn, and one of the ideas finally led to the solution." It turned out, however, that van der Waerden could not demonstrate the result for two colors without simultaneously demonstrating it for an arbitrary number of colors.

For his proof, van der Waerden employed a special form of mathematical induction. The usual form, known as single induction, has two steps. First, one shows that the result holds for some small value, such as two. Second, one proves that if the result holds for any value then it holds for the next larger value. This implies that it holds for three, four and so on. The results fall like an infinite set of dominoes.

Van der Waerden employed a more subtle, double induction to prove Ramsey's theorem for arithmetic progressions. He assumed that for any fixed number of colors there was a number *n* such that if each integer from 1 through *n* were printed in one of these colors, then there would be a monochromatic arithmetic progression of, say, 10 terms. He could then deduce that for any fixed number of colors, there was a number *m* such that if each integer from 1 through *m* were printed in one of these colors, then there would be a monochromatic arithmetic progression of 11 terms. In general, he showed that knowing the result for k terms and all numbers of colors implies the result for $k+1$ terms and all numbers of colors.

Once van der Waerden had arrived at that stage in the proof, he had only to demonstrate that the result does hold for some small value of k. If the number of integers is one more than the number of colors, then there are always two integers that have the same color. These two integers form an arithmetic progression of two terms. Thus, if the number of integers is one more than the number of colors, there is always a monochromatic arithmetic progression of two terms. The infinite set of dominoes with two terms now pushes over the infinite set with three terms, which in turn pushes over the infinite set with four terms and so on [*see box on this page*].

Having proved Ramsey's theorem for arithmetic progressions, van der Waerden applied his knowledge to the following problem: What is the smallest value of *n* that will guarantee a monochromatic arithmetic progression of, say, 10 terms if each integer from 1 through *n* is printed arbitrarily in one of two colors? The best answer that van der Waerden could find was so large that it cannot be written in conventional notation. It was larger than a billion, larger than 10 to the power of a billion.

In fact, in order to express his result, mathematicians rely on a sequence of functions known as the Ackermann hierarchy. The first function in the hierarchy is simply called DOUBLE(x). As the name implies, the function doubles the number x. Therefore, DOUBLE(1) equals 2, and DOUBLE(50) equals 100. The second function, EXPONENT(x), can be expressed as 2 to the power of x, and therefore, EXPONENT(3) equals 8. We also can describe EXPONENT in terms of DOUBLE. To find EXPONENT (3), for instance, we double 1, then double the result, then double the result again. In fact, each function in the Ackermann hierarchy is defined in terms of its predecessor.

Hence, the third function in the hierarchy, TOWER(x), can be expressed using EXPONENT. TOWER(3), for example, is 2 to the power of 2 to power of 2, which equals 2 to the power of 4, or 16. TOWER(x) is sometimes written as a tower of exponents

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$$

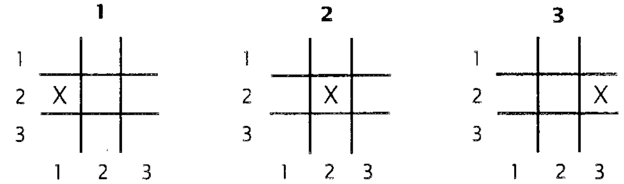
where there are x number of 2's in the

Ramsey Theory and Tic-Tac-Toe

In 1926 Bartel L. van der Waerden proved that if *n* is a sufficiently large integer and if each integer 1 through *n* is printed arbitrarily in one of two colors, then there is always a monochromatic progression with a certain number of terms. In 1963 Alfred W. Hales and Robert I. Jewett found what has proved to be the essence of van der Waerden's theorem while investigating the game tic-tac-toe. Although the classic three-on-a-side tic-tac-toe can get tiresome, four-on-a-side tic-tac-toe in three dimensions is quite challenging. The board for the three-dimensional game has 64 cells arranged in a cube. Players alternately fill the cells with naughts and crosses until one player wins by occupying four cells in a line. Two- and three-dimensional tic-tac-toe sometimes end in a tie. But what about higher-dimensional games? Is a player ever guaranteed to win in some *n*-dimensional *k*-in-a-row version of tic-tac-toe?

Hales and Jewett showed that if the dimension *n* is large enough, one can always find a *k*-in-a-row version that *never* ends in a tie. For instance, no matter how the naughts and crosses are arranged on a three-dimensional three-in-a-row version, either three naughts will occupy a line or three crosses will occupy a line.

Van der Waerden's theorem can be derived from the Hales-Jewett result by employing a transformation that converts lines of tic-tac-toe into arithmetic progressions. Consider a game of three-in-a-row tic-tac-toe in three dimensions.



The coordinates for the crosses in this winning combination are 121, 222 and 323, which form an arithmetic progression. It can be shown that any winning combination transformed by this method will yield an arithmetic progression.

tower. Yet even the TOWER(x) function does not increase rapidly enough to describe van der Waerden's result.

The next function, informally known as WOW(x), is found by beginning at 1 and applying the TOWER function x times. Therefore,

WOW(1) = TOWER(1) = 2
WOW(2) = TOWER(2) = 4
WOW(3) = TOWER(4) = 65,536

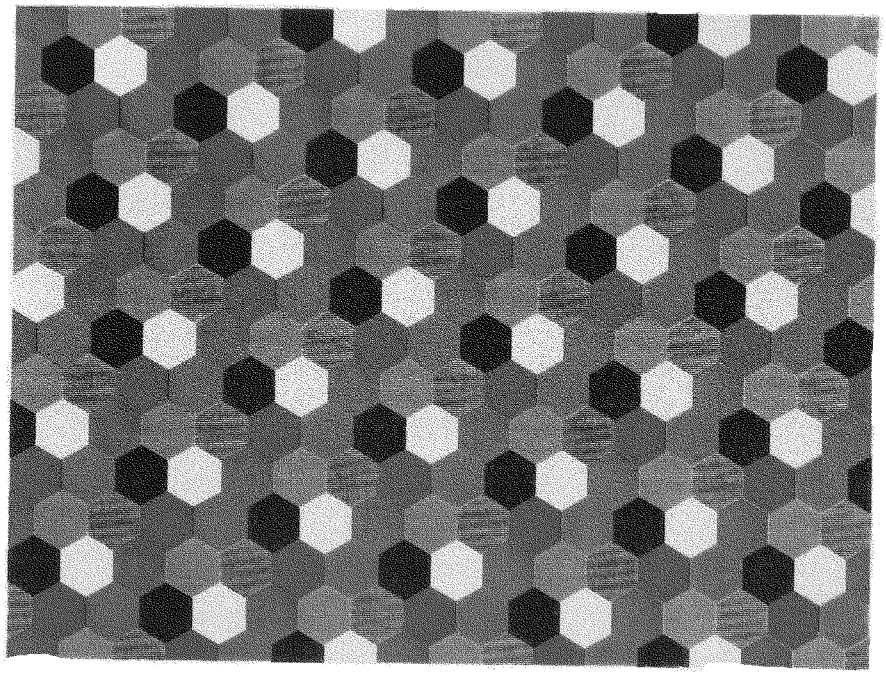
To find WOW(4), we need to compute TOWER(65,536). To do this, we begin at 1 and apply EXPONENT 65,536 times. Even applying EXPONENT just five times gives $2^{65,536}$, a number whose digits would fill two pages of this magazine. In fact, if a number filled every page of every book and every memory bank of every computer, it would still be incomparable to WOW(4).

Yet to give van der Waerden's result, we must define a function that grows even faster. The function ACKERMANN (x) is defined by the sequence DOUBLE (1), EXPONENT(2), TOWER(3), WOW(4) and so on. ACKERMANN(x) eventually dominates all of the functions of the hierarchy. Van der Waerden's proof gave the following quantitative result: if the integers 1, 2,..., ACKERMANN(k) are two-colored, then there is always a monochromatic arithmetic progression of k terms.

It seemed preposterous that such enormous numbers could come out of such an innocent statement involving only arithmetic progressions. Over the years many mathematicians attempted to improve the proof of van der Waerden. As the failures mounted, the idea began to gain support that a double induction and the corresponding ACKERMANN function were necessary features in any proof of van der Waerden's theorem. Increasingly, logicians tried to supply arguments that this indeed was so.

In 1987, however, Israeli logician Saharon Shelah of the Hebrew University in Jerusalem achieved a major breakthrough. Shelah is widely regarded as one of the most powerful problem solvers in modern mathematics. He broke through the ACKERMANN barrier to show the following: if the integers 1, 2,...,WOW(k) are two-colored, then there must always be a monochromatic arithmetic progression of k terms.

Despite his background, Shelah's proof uses no tools from mathematical logic whatsoever. His proof employs only elementary (but highly ingenious) mathematical ideas. Written out in full, the proof is perhaps four pages long, and most experts consider it clearer than van der Waerden's original proof.



CONCEPTS in Ramsey theory can be applied to problems in geometry such as this hexagon puzzle. If the sides of the hexagons are all .45 unit long (the unit is arbitrary), then two points within a hexagon are at most .9 unit apart. Each hexagon is shaded with one of seven colors so that no two hexagons of the same color are less than 1.19 units apart. No two points of the same color are precisely one unit apart. No one has been able to determine whether or not the plane can be shaded with six colors so that no two points of the same color are precisely one unit apart.

Most important, he avoids the double induction. He fixes the number of colors at two (or any particular number) and then proves a simple induction: if the result holds for progressions of k terms then it also holds for progressions of (k + 1) terms.

Mathematicians are now poring over Shelah's proof to see if it can in fact be further improved to give a TOWER or even an EXPONENT function for van der Waerden's theorem. One of us (Graham) has offered a reward of $1,000 for a proof (or disproof) that for each number k, if the numbers 1, 2,..., TOWER(k) are two-colored, then a monochromatic arithmetic progression of k terms must be formed.

The work of Ramsey, Erdős, van der Waerden and many others established the fundamentals of Ramsey theory. Yet workers have only begun to explore the implications of the theory. It suggests that much of the essential structure of mathematics consists of extremely large numbers and sets, objects so huge that they are difficult to express, much less understand.

As we learn to handle these large numbers, we may find mathematical relations that help engineers to build large communications networks or help scientists to recognize patterns in large-scale physical systems. Today we can easily recognize the constellations in the night sky as a consequence of Ramsey theory. What patterns may we find in sets that are ACKERMANN(9) times larger?

FURTHER READING

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